

Theoretical Physics

Preprint

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Heiko Bauke* and David Sherrington

Rudolf Peierls Centre for Theoretical Physics, University of Oxford,
1 Keble Road, Oxford, OX1 3NP, United Kingdom

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Rudolf Peierls Centre for Theoretical Physics, University of Oxford, 1 Keble Road, Oxford, OX1 3NP, United Kingdom

(Dated: May 25, 2007)

In this contribution we introduce *local attachment* as an universal network-joining protocol for peer-to-peer networks, social networks, or other kinds of networks. Based on this protocol nodes in a finite-size network dynamically create power-law connectivity distributions. Nodes or peers maintain them in a self-organized statistical way by incorporating local information only. We investigate the structural and macroscopic properties of such local attachment networks by extensive numerical simulations, including correlations and scaling relations between exponents. The emergence of the power-law degree distribution is further investigated by considering preferential attachment with a nonlinear attractiveness function as an approximative model for local attachment. This study suggests the local attachment scheme as a procedure to be included in future peer-to-peer protocols to enable the efficient production of stable network topologies in a continuously changing environment.

PACS numbers: 89.20.Ff, 89.75.Da, 89.75.Fb

1. Introduction

Information networks and other kinds of networks are an ubiquitous element of modern infrastructures. These networks can be established by actual hardware installations as network cables and power lines or may be of conceptual notion only, as for example in the global network of web pages or in *ad hoc* peer-to-peer networks.

Networks that are found in nature or in technical or socio-economic systems differ in their statistical properties remarkably from random networks of Erdős-Rényi type. Therefore, much attention has been paid to trying to gain a better understanding of these networks [1–3].

In this paper we discuss a network model which has been inspired by *ad hoc* peer-to-peer networks [4, 5]. These kinds of networks provide an approach to computer connectivity that is complementary to traditional client-server-architectures. Peer-to-peer networks are designed to establish a reliable service in a self-organized way without any dependence on a central instance. Their structural and dynamic properties emerge solely from local decisions of individual peers. The lack of a single point of failure makes peer-to-peer systems potentially more robust than client-server systems, especially in unreliable constantly changing environments. The peer-to-peer paradigm became popular through file sharing systems such as Napster or Gnutella. But file sharing is just the tip of the iceberg, new peer-to-peer applications such as internet telephony, distributed computing, and applications in mobile environments are an active field of research [4].

In an empirical study of the Gnutella network [6] it has been observed that peer-to-peer networks have a nontrivial degree distribution in form of a power-law. In fact, networks with power-law degree distribution $p(k) \sim k^{-\gamma}$ have properties which might be useful in some peer-to-peer applications. In power-law networks one can apply local search strategies that scale sub-linearly with the number of nodes N [7, 8],

whereas the search time scales linearly with the system size in networks of Erdős-Rényi type. If the exponent of the degree distribution is in the range $2 < \gamma < 3$, the diameter¹ grows very slowly, namely proportionally to $\ln \ln N$ [9]. Furthermore, these networks are very robust against random node failures [10, 11].

Several growth models have been proposed, that are able to produce networks with a power-law degree distribution [1–3]. On the other hand, in peer-to-peer networks nodes disappear from the network and others (or the same later) join the network regularly and over long time scales peer-to-peer networks have essentially a constant size. In straightforward generalizations of growth models to constant size network models the power-law degree distribution is not necessarily preserved. In fact, the classical growth model of preferential attachment by Barabási and Albert [12] fails to produce power-law networks if nodes enter and disappear at the same rate, see [13] and section 3.1.

Only a few network evolution models have been proposed, that are able to produce power-law networks of fixed-size. For example in [13] an active rewiring is introduced to recover the power-law of preferential attachment under node deletion. The non-growth model [14] gives an unusual small exponent $\gamma = 1$, by incorporating random node deletion and a non-local node copy operation.

Here we introduce a network model, called *local attachment* model. In this model node attachment is a genuine local operation. It is able to emulate peer-to-peer practice and also to maintain a power-law degree distribution for both growing networks and evolving networks of constant size. Our local attachment scheme can be used to implement new or improve existing (unstructured) peer-to-peer protocols.

This paper is organized as follows: In section 2 we introduce the local attachment mechanism as a network-joining protocol. It might be utilized to organize so-called constant

*E-mail: heiko.bauke@physics.ox.ac.uk

¹ The diameter of a network refers to the longest shortest path between any two nodes in a network.

size networks under churn, where nodes leave and join a network constantly, or in the context of a pure growth model. Section 3 investigates local attachment in constant size networks, while section 4 focuses on a local attachment growth model.

2. The local attachment mechanism

The mechanism that generates the first connection of a new node to an existing peer-to-peer network (*bootstrapping*) is a crucial detail of the design of a peer-to-peer protocol. Once a single connection to the network has been established, the addition of further connections becomes easy. Remarkably, Gnutella, probably the best documented peer-to-peer protocol [4, 15], does not define the bootstrap process in detail.

One of the different bootstrap procedures that are actually implemented by existing software works as follows: It is assumed that a peer who wants to connect to the network knows already another peer who is likely a member of the network at this moment. He might remember this peer from connections in the past or become aware of such a peer by means of a directory service that knows some members of the network. In the case of Gnutella this service is called *GWebCache*. After a new peer has got its first connection to the network he might ask his neighbor about its neighbors. In the jargon of the Gnutella protocol this is called *pong caching*. The new node may establish connections to these peers as well, until he owns a reasonable number of connections.

Real world networks cannot grow forever, after a certain time of growth they maintain a finite size. In fact, a crucial feature of nodes in peer-to-peer networks is that they join the network only for a certain time, which is much shorter than the lifetime of the network. If nodes can leave the network the process is characterized as *under churn*. Nodes must enter and leave the network at the same rate, at least on average. Otherwise the network would die out or grow *ad infinitum*. The rate at which nodes leave (and join) the network is referred to as the *churn rate*.

This leads us to our model of local attachment of networks under churn. Initially an arbitrary network of N nodes is given. At each time step t a randomly chosen node and all edges incident to this node are deleted from the network and a new node $u = t$ enters the network by introducing m connections to targets in the existing network. Selecting a target from the current network by local attachment is a two-step procedure.

1. First we pick a still-remaining old node v_1 randomly with uniform probability from the set of nodes with non-zero degree. The new node will not be connected to this node; to do so would lead to an exponential degree distribution [12, 16].
2. Instead we assume that the new node u is able to explore the neighborhood of v_1 and will connect itself to a node v_2 uniformly randomly chosen among the neighbors of v_1 .

This two-step procedure is iterated until the degree of the new node equals m . Note that a new node incorporates only minute

information about the network topology to establish its connections; it is a genuine local process.

If nodes leave the network at a lower rate than others enter the network or nodes do not leave at all, local attachment leads to a growth model. In the growth version of local attachment at each time step t a new node $u = t$ enters the network by connecting to m existing nodes. Each connection is established by the two-step procedure described above.

Similar approaches to growth models where the attachment of new nodes to the network is based on local rules have been considered by [17–19]. In the random walk model [17] new nodes are added to the network by establishing a directed edge to some randomly chosen node. This model resembles a random walk on the network because each time when an edge is created to a vertex in the network then with some probability p an edge is also created to one of the nearest neighbors of this vertex. The walk stops with probability $(1 - p)$ and the next new node is added to the network. In the model considered in [18] new nodes are connected to both ends of a randomly chosen edge by two undirected links. This leads to networks with very high clustering. In [19] the authors consider a growth process of directed networks which results in networks with a simple treelike topology. At each time each node has an out-degree of one but an arbitrary in-degree. In this model each new node u is either connected to a randomly chosen node v or to the unique neighbor of v . Both processes happen with some probability $(1 - p)$ and p , respectively. A similar model has been introduced in the context of the world wide web [20].

Although local attachment has been motivated in the context of peer-to-peer networks, it may model other kinds of networks as well. Social networks where old members introduce new members to their neighbors are an obvious example.

3. Local attachment networks under churn

3.1. Macroscopic quantities

If nodes enter and leave a network at the same rate, the microscopic structure of the network is constantly changing. But there are macroscopic quantities that evolve asymptotically to limiting values (in a statistical sense), e. g. the degree distribution or the number of edges. If old nodes leave randomly and new nodes enter the network by making m connections to the existing network, the number of edges in the network will fluctuate around $mN/2$ independently of the details of the underlying microscopic dynamics of the network-joining protocol. But in general, other macroscopic quantities depend on the details of the way that new nodes connect to the network.

Barabási and Albert introduced in [12] the mechanism of *preferential attachment*, which yields in growing networks (no nodes are removed) a power-law degree distribution $p(k) \sim k^{-\gamma}$ with exponent $\gamma = 3$. The attachment mechanism assigns explicitly to each node an attractiveness $A(k)$ that is proportional to its degree k . Whenever a new node u is added

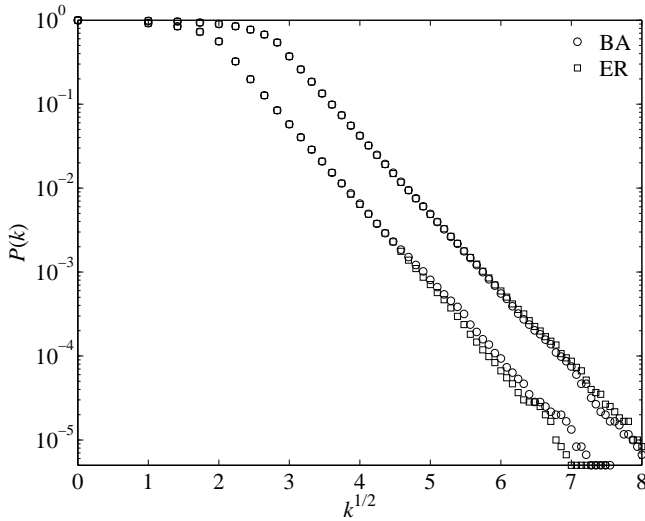


Figure 1: Cumulative degree distribution $P(k) = \sum_{i \geq k} p(i)$ for evolving networks of constant size undergoing preferential attachment under churn with $N = 20000$ nodes and a mean degree m . Randomly chosen nodes disappear from the network and new nodes enter the network at the same rate. New nodes connect to m old nodes via preferential attachment (lower line $m = 4$, upper line $m = 8$). Initial networks were given either by an Erdős-Rényi (ER) network or by a Barabási-Albert (BA) network.

to the growing network it attaches to m old nodes v which are chosen randomly with probability

$$p_{\text{add}}(v) = \frac{A(k_v)}{\sum_i A(k_i)} = \frac{k_v}{2mt}, \quad (1)$$

where k_i is the degree of node i and $t = N$ is the current number of nodes in the network. This growth process starts from some small initially-given network, e. g. $m + 1$ fully connected nodes, and the quantity t may be interpreted as the time that passed since the growth process started.

However, the power-law degree distribution of the Barabási-Albert model is not preserved if the network is under churn. If new nodes enter the network via preferential attachment but random nodes leave the network at the same rate [13], the degree distribution converges independently of the initial network to a distribution with a tail that falls faster than any power-law. In fact, we find numerically that the tail of its cumulative degree distribution $P(k) = \sum_{i \geq k} p(i)$ falls as $\mathcal{O}(\exp(-k^{1/2}))$ and therefore $p(k) \sim \mathcal{O}(\exp(-k^{1/2}))$, see Figure 1. To maintain the power-law degree distribution under node deletion a deletion-compensation protocol was introduced by [13] in which, if a node has lost a connection, it initiates on average n new connections by choosing nodes by preferential attachment. This allows for any given deletion rate to tune the power-law exponent of the degree distribution to be anywhere in $(2, \infty)$ by varying the average number of compensatory edges for each deleted edge n .

On the contrary, if new nodes enter the network by local attachment instead of preferential attachment, the deletion-compensation protocol becomes unnecessary. In fact, if new

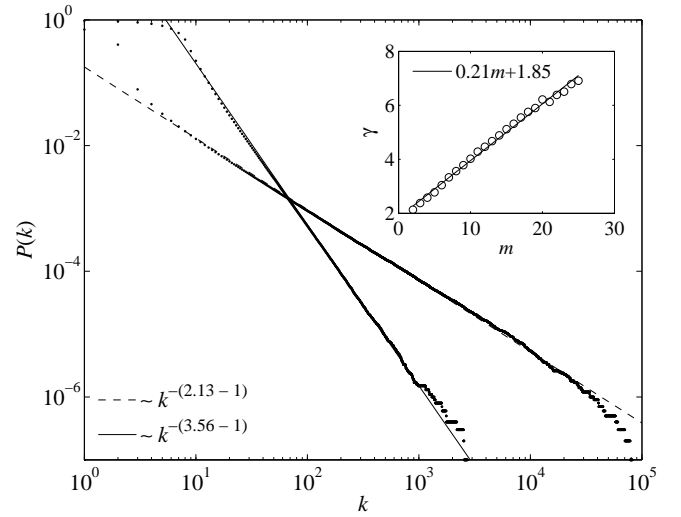


Figure 2: Cumulative degree distribution $P(k) = \sum_{i \geq k} p(i)$ for a network under churn evolving via local attachment and a fit of this distribution to a power-law $P(k) \sim k^{-(\gamma-1)}$, which corresponds to $p(k) \sim k^{-\gamma}$. Randomly chosen nodes disappear from the network and new nodes enter the network at the same rate. New nodes connect to m old nodes via local attachment. In these simulations the initial networks were Erdős-Rényi with $mN/2$ edges, but the steady-state power-law behavior is independent of the details of the starting state. Results in the main figure are shown for $m = 2$ (dashed line), and $m = 8$ (solid line) for $N = 100000$ nodes. In the inset: Exponent γ of the power-law degree distribution $p(k)$ as a function of m . Distributions and exponents have been determined by averaging over 100 networks.

nodes are inserted via local attachment, the degree distribution exhibits a power-law even if nodes leave and enter at the same rate, see Figure 2. Numerically we find that the exponent γ of the degree distribution $p(k)$ grows linearly with the mean degree m of the network. This allows one to tune the exponent γ by choosing an appropriate mean degree m .

For a peer-to-peer network it is desirable that the network stays connected, even if nodes leave (and enter) the network constantly. Of course, the more edges a network contains (parameter m large), the smaller the probability that the network falls apart into more than one connected component. Numerically we find that networks that evolve under the proposed dynamics consist of a large connected component and isolated nodes or small components of size $\mathcal{O}(1)$, see Figure 3. The number of nodes outside the largest connected component is rather small compared to the size of the largest component. If the mean degree m is greater than or equal to five, less than 10 % of the nodes are not part of the large component.

3.2. Node attractiveness and degree correlations

If a new node u establishes an edge to the network by our local attachment rule, the connection will be made to node v_2 with

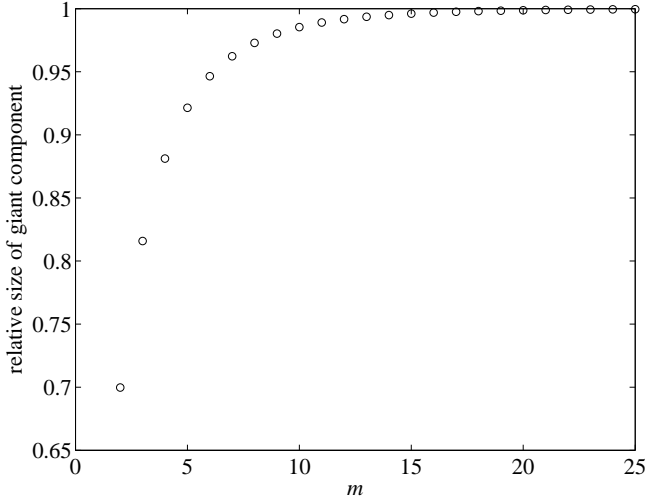


Figure 3: Mean size of the largest connected component of local attachment networks under churn of $N = 100\,000$ nodes but with different mean degree m .

probability

$$p_{\text{add}}(v_2) = \frac{1}{N} \sum_{v_1 \in \{\text{nn } v_2\}} \frac{1}{k_{v_1}}, \quad (2)$$

where the sum runs over the neighbors of node v_2 and k_{v_1} denotes the degree of node v_1 . The probability (2) can be rephrased in terms of a node attractiveness $A(v_2)$ by

$$p_{\text{add}}(v_2) = \frac{A(v_2)}{\sum_w A(w)} \quad \text{with} \quad A(v_2) \sim \sum_{v_1 \in \{\text{nn } v_2\}} \frac{1}{k_{v_1}}. \quad (3)$$

Note that in general the attractiveness of a node is determined uniquely up to a multiplicative constant only. Because of the two-step nature of local attachment the attractiveness of a node depends on its own degree and the degrees of its nearest neighbors, too, and therefore correlations between the degrees of neighboring nodes are important.

In uncorrelated networks local attachment is equivalent to preferential attachment, in the sense that a node with degree k has *on average* an attractiveness proportional to its degree. Let $\langle A(k) \rangle$ be the average attractiveness of a node with degree k and $p_{\text{nn}}(i|k)$ the probability that the degree of a nearest neighbor of a node with degree k equals i , then

$$\langle A(k) \rangle = \frac{\sum_{i_1, \dots, i_k=1}^{\infty} \left(\prod_{j=1}^k \frac{1}{i_j} \right) \prod_{h=1}^k p_{\text{nn}}(i_h|k)}{\sum_{i_1, \dots, i_k=1}^{\infty} \prod_{h=1}^k p_{\text{nn}}(i_h|k)}, \quad (4)$$

where i_h is the degree of the h th neighbor of a node with degree k .

For uncorrelated networks $p_{\text{nn}}(i|k)$ does not depend on k and equals the ordinary degree distribution $p(i)$ and thus we get

$$\langle A(k) \rangle = k \frac{\sum_{i=1}^{\infty} p(i)/i}{1 - p(0)}. \quad (5)$$

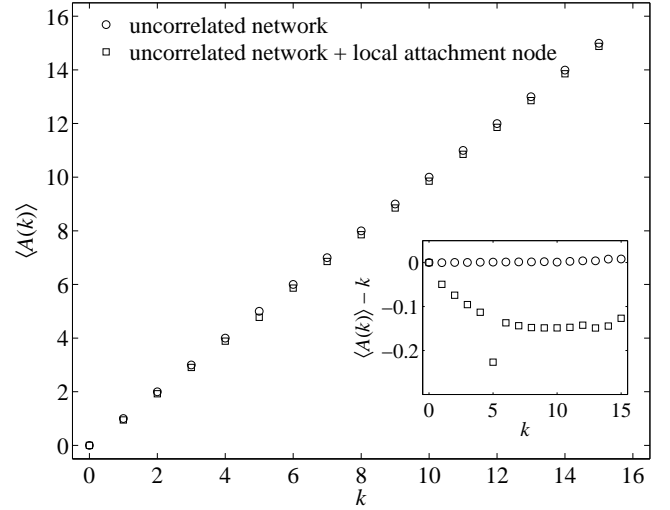


Figure 4: Average attractiveness $\langle A(k) \rangle$ as a function of the degree for uncorrelated networks and for networks in which a single extra node has been added to an uncorrelated network via $m = 5$ new edges. The uncorrelated networks have $N = 40$ nodes and each edge is present with probability m/N independently. Results have been averaged over 10^6 networks.

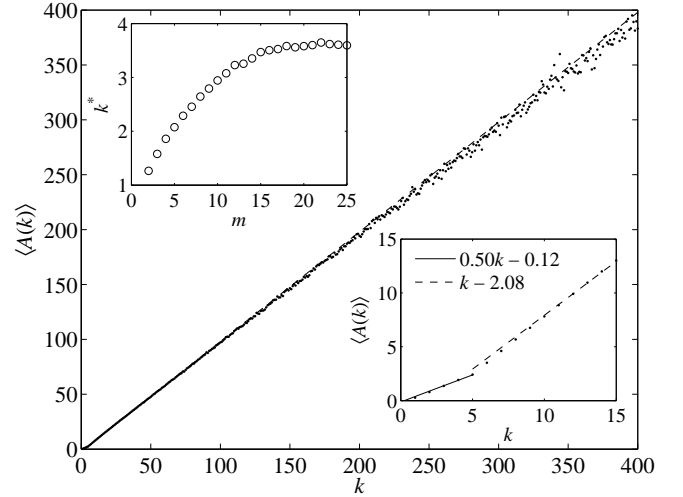


Figure 5: Mean attractiveness of a node as a function of its degree in a local attachment network under churn for $m = 5$. The inset in the right lower corner shows a detail of this distribution for small degrees. After an arbitrary rescaling the attractiveness $\langle A(k) \rangle$ is approximately given by $\langle A(k) \rangle \approx k - k^*$ for degrees $k \geq m$. The offset k^* as a function of the mean degree of the network is shown in the inset in the upper left.

After an arbitrary rescaling (5) equals the attractiveness of the Barabási-Albert model. But note that in Barabási-Albert networks the attractiveness $A(k) \sim k$ is imposed and these networks have structural properties that are different from uncorrelated networks.

On the other hand, the local attachment mechanism *does* induce correlations and as a consequence the average attractiveness $\langle A(k) \rangle$ does not follow the linear law (5). This can be illustrated by a simple numerical experiment.

In this experiment a single node enters an uncorrelated Erdős-Rényi network by the local attachment rule. For each network the attractiveness as a function of the degree has been determined *before* node attachment and *afterward*. The attractiveness was averaged over a large number of networks and has been normalized arbitrarily such that $\langle A(k) \rangle$ grows with a slope of one for $k > m$, see Figure 4. The effect of the correlations induced by a single local attachment node is minute but clear, $\langle A(k) \rangle$ becomes a nonlinear function. In fact, it is piecewise linear with different slope for $k < m$ and $k > m$.

The more nodes enter the network by local attachment the stronger the degree-degree-correlations, but on the other hand random node removal destroys correlations. The competition between these two processes leads finally to a dynamical steady state. Empirically we find that in this steady state the average attractiveness $\langle A(k) \rangle$ is (after an arbitrary normalization) approximately given by the piecewise linear function

$$\langle A(k) \rangle = \begin{cases} \left(1 - \frac{k^*}{m}\right) k & \text{if } 0 \leq k \leq m \\ k - k^* & \text{if } k \geq m, \end{cases} \quad (6)$$

see Figure 5.

Other quantities that are sensitive to degree-degree-correlations are the distribution

$$\langle k_{nn}(k) \rangle = \sum_i i p_{nn}(i|k) \quad (7)$$

of the mean degree of the neighboring nodes of a node with degree k and the distribution

$$\Delta k_{nn}(k) = \sqrt{\sum_i (i - \langle k_{nn}(k) \rangle)^2 p_{nn}(i|k)} \quad (8)$$

of the standard deviation of the degree distribution of the neighboring nodes of a node. In uncorrelated networks both quantities do not depend on the degree k .

However, our local attachment mechanism does induce correlations between low degree nodes and their neighbors and the distributions (7) and (8) are not flat. As a consequence of the constant deletion and addition of nodes the distributions (7) and (8) of local attachment networks under churn can be divided into two different regimes. At the level of large k degree-degree-correlations are averaged out and the distribution $\langle k_{nn}(k) \rangle$ is almost constant and roughly equals the mean degree of the whole network m , independently of the system size N , see Figure 6. Furthermore, the distribution of the degrees of neighboring nodes is rather narrow and therefore $\Delta k_{nn}(k) < \langle k_{nn}(k) \rangle$. On the other hand, low degree nodes attach preferably to high degree nodes and the degree $\langle k_{nn}(k) \rangle$ is much larger than the mean degree of the whole network m . One says, the network shows *disassortative* correlations. The distribution of the degrees of neighboring nodes is very broad in this regime and at fixed k both quantities $\langle k_{nn}(k) \rangle$ and $\Delta k_{nn}(k)$ grow with the system size N .

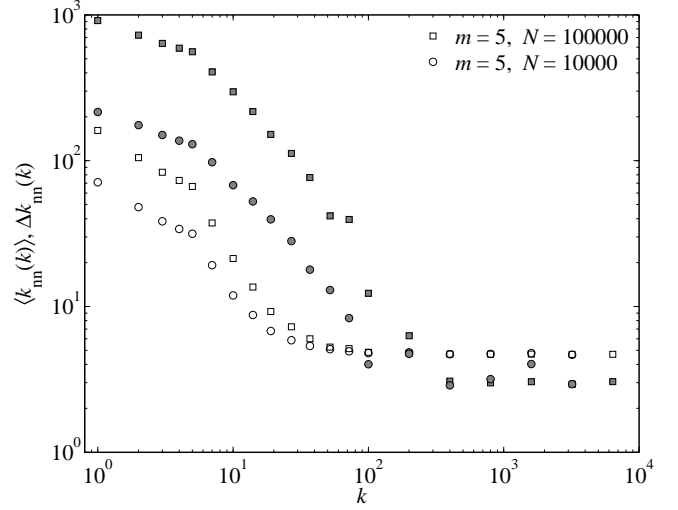


Figure 6: Distribution $\langle k_{nn}(k) \rangle$ of the mean degree of the neighboring nodes (open symbols) and the distribution $\Delta k_{nn}(k)$ of the standard deviation of the degree distribution of the neighboring nodes (closed symbols) of a node with degree k for nodes in local attachment networks under churn.

3.3. The degree distribution: theoretical considerations

In the last section we have shown that correlations are an important feature of local attachment networks under churn. A full mathematical description of these networks has to take them into account. This is a nontrivial task. However, the power-law degree distribution of local attachment networks under churn can be explained qualitatively by a self-consistent description, if only correlations on the level of the nonlinear mean attractiveness (6) are taken into account.

3.3.1. Preferential attachment with the nonlinear attractiveness

For this description we consider a preferential attachment model under churn with the nonlinear attractiveness

$$A(k) = \begin{cases} \left(1 - \frac{k^*}{m}\right) k & \text{if } 0 \leq k \leq m \\ k - k^* & \text{if } k \geq m, \end{cases} \quad (9)$$

as an approximative model for our local attachment model. Note that in this nonlinear preferential attachment model the attractiveness (9) is imposed explicitly, while in the local attachment model the nonlinear mean attractiveness (6) emerges implicitly from the dynamics of the model.

Let $D(i, t)$ be the probability that a node that entered the network at time i is still present at time $t \geq i$. Node removal does not depend on time t nor on the removal of other nodes and therefore $D(i, t)$ depends only on the age $t' = t - i$ of a node. For $N \rightarrow \infty$ and $(t - i) \rightarrow \infty$ such that $(t - i)/N \rightarrow \text{const.}$

the probability $D(i, t)$ becomes

$$D(i, t) = D(t') = \left(1 - \frac{1}{N}\right)^{t'} \approx e^{-\frac{t'}{N}}. \quad (10)$$

For the calculation of the degree distribution $p(k)$ we adopt a continuous variable approximation as used in [13, 21]. Let $\langle k(i, t) \rangle$ be the mean degree (ensemble average over different network realizations) at time $t \geq i$ of a node that has entered the network at time i and has not yet disappeared from the network. In the framework of a continuous approximation the evolution of $\langle k(i, t) \rangle$ is given by

$$\frac{\partial \langle k(i, t) \rangle}{\partial t} = m \frac{A(\langle k(i, t) \rangle)}{S(t)} - \frac{\langle k(i, t) \rangle}{N} \quad (11)$$

with the initial condition $\langle k(i, i) \rangle = m$, where the quantity $S(t)$ is given by

$$S(t) = \int_0^t D(i, t) A(\langle k(i, t) \rangle) di. \quad (12)$$

If the degree distribution $p(k)$ of a fixed-size network under churn converges for $t \rightarrow \infty$ to a limiting distribution, then $S(t)$ must converge to a constant. This motivates us to introduce the quantity

$$s = \lim_{t \rightarrow \infty} \frac{S(t)}{N} = \frac{1}{N} \lim_{t \rightarrow \infty} \int_0^t D(i, t) A(\langle k(i, t) \rangle) di. \quad (13)$$

After the network has reached some time-independent degree-distribution s is given by the mean attractiveness of the network

$$s = \int_0^\infty p(k) A(k) dk. \quad (14)$$

The mean attractiveness s depends on the unknown degree distribution $p(k)$, but we can make some general statements about s by taking into account the imposed attractiveness $A(k)$. If we assume that $A(k)$ is given by (9) then

$$\begin{aligned} s &= \int_0^m p(k) \left(1 - \frac{k^*}{m}\right) k dk + \int_m^\infty p(k) (k - k^*) k dk \\ &= m - k^* \left(\int_0^m \frac{k}{m} p(k) dk + \int_m^\infty p(k) dk \right), \end{aligned} \quad (15)$$

where m is the mean degree of the network. From (15) follows the inequality $m - k^* < s < m$.

The attractiveness (9) is piecewise linear, therefore we have to distinguish two cases for the solution of (11). For $\langle k(i, t) \rangle < m$ equation (11) becomes

$$\begin{aligned} \frac{\partial \langle k(i, t) \rangle}{\partial t} &= \frac{m}{sN} \left(1 - \frac{k^*}{m}\right) \langle k(i, t) \rangle - \frac{1}{N} \langle k(i, t) \rangle \\ &= \frac{m - k^* - s}{sN} \langle k(i, t) \rangle. \end{aligned} \quad (16)$$

Because $(m - k^* - s) < 0$, this equation has exponentially decaying solutions for any initial condition $\langle k(i, t_0) \rangle = k_0$ with $t_0 \geq i$,

$$\langle k(i, t) \rangle = k_0 e^{\frac{m - k^* - s}{s} \frac{t - t_0}{N}}. \quad (17)$$

On the other hand, for $\langle k(i, t) \rangle > m$ we find the evolution equation

$$\begin{aligned} \frac{\partial \langle k(i, t) \rangle}{\partial t} &= m \frac{\langle k(i, t) \rangle - k^*}{sN} - \frac{1}{N} \langle k(i, t) \rangle \\ &= \frac{m - s}{sN} \langle k(i, t) \rangle - \frac{mk^*}{sN}, \end{aligned} \quad (18)$$

which has the solution

$$\langle k(i, t) \rangle = \frac{mk^*}{m - s} + \left(k_0 - \frac{mk^*}{m - s}\right) e^{\frac{m - s}{s} \frac{t - t_0}{N}} \quad (19)$$

for the initial condition $\langle k(i, t_0) \rangle = k_0$ with $t_0 \geq i$. The exponent $(m - s)/s$ in (19) is positive, but the prefactor $[k_0 - mk^*/(m - s)]$ might be positive or negative depending on the initial condition, which corresponds to exponentially growing or shrinking $\langle k(i, t) \rangle$.

New nodes enter the network by establishing m new edges at time $t = i$ and thus the solution (19) reads

$$\langle k(i, t) \rangle = \frac{mk^*}{m - s} + \frac{m(m - k^* - s)}{m - s} e^{\frac{m - s}{s} \frac{t - i}{N}}. \quad (20)$$

Because $(m - k^* - s) < 0$, this solution predicts an exponentially fast decay of $\langle k(i, t) \rangle$ for (young) nodes with degree $\gtrsim m$. The initial condition $\langle k(i, i) \rangle = m$ is just on the borderline, but in both cases our calculations predict a decaying mean degree $\langle k(i, t) \rangle$.

Figure 7 shows that $\langle k(i, t) \rangle$ is indeed decaying, but only for young nodes. In fact, if a node has reached a sufficient large age $\langle k(i, t) \rangle$ grows exponentially. The crossover from decaying to exponentially growing $\langle k(i, t) \rangle$ is driven by statistical fluctuations among the degrees of individual nodes. Because of these statistical fluctuations individual nodes can gain some degree k_0 such that $[k_0 - mk^*/(m - s)] > 0$. In this case (19) predicts that the degree of such nodes will grow exponentially fast (on average).

This interpretation is compatible with our numerical experiments, as Figure 7 shows. By construction, nodes enter the network by establishing exactly m connections. Then the degree of young nodes shrinks on average but simultaneously the distribution of the degree of nodes of the same age gets broader with increasing age. As a consequence, a certain fraction of nodes can reach a critical degree, such that exponential growth can take off.

According to (20) the exponent of the exponential growth is given by

$$\beta = \frac{m - s}{s}. \quad (21)$$

We may use the degree distribution and $\langle k(i, t) \rangle$ to determine numerical values for s (see equation (15)) and β independently. Our results confirm that in the case of preferential

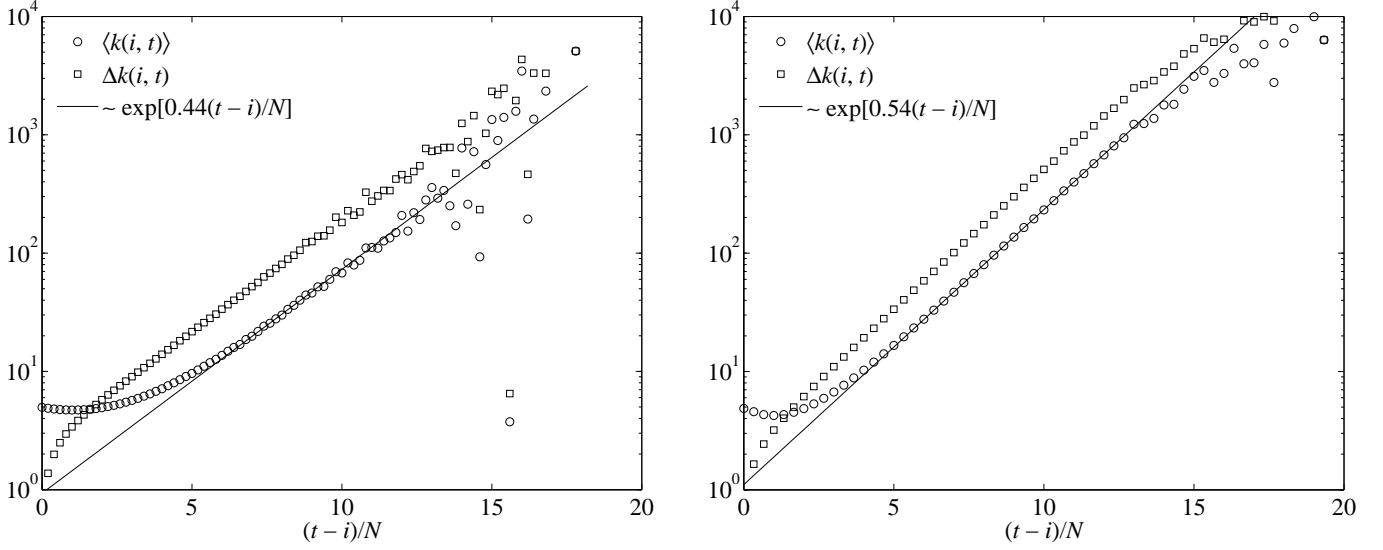


Figure 7: Mean degree $\langle k(i, t) \rangle$ of a node and the standard deviation $\Delta k(i, t)$ of the distribution of the degree as a function of its age $(t-i)$. The diagrams show results for two different kinds of networks under churn, a preferential attachment network with a nonlinear attractiveness (9) (left) and a local attachment network (right). Both networks have $N = 100000$ nodes, a mean degree $m = 5$, and the parameter of the attractiveness function equals $k^* = 2.077$.

attachment networks with a nonlinear attractiveness (9) the relation (21) is approximately fulfilled, see Table I.

Note that in the nonlinear preferential attachment model of constant size networks under churn the mean degree of a node grows exponentially with its age $t' = t - i$ proportionally to $e^{\beta t'/N}$, whereas in growth models that yield a power-law degree distribution this quantity grows proportional to some power-law $(t/i)^\beta$ [22].

The degree distribution $p(k)$ follows from the mean degree $\langle k(i, t) \rangle$. Our numerical findings show that for sufficiently large ages $t' = t - i$ the distribution $\langle k(i, t) \rangle$ is given by an exponential law

$$\langle k(i, t) \rangle = \langle k(t') \rangle = c_1 + c_2 e^{\frac{m-s}{s} \frac{t'}{N}}. \quad (22)$$

Table I: Characteristic quantities of preferential attachment networks with a nonlinear attractiveness (9) under churn and local attachment networks under churn, see text for an explanation of m , k^* , s , β , and γ .

m	k^*	s	β	$(m-s)/s$	γ	$m/(m-s)$	$1/(\gamma-1)$
preferential attachment networks with nonlinear attractiveness							
3	1.58	1.92	0.551	0.559	2.85	2.79	0.540
4	1.86	2.66	0.491	0.503	3.04	2.99	0.490
5	2.08	3.44	0.438	0.453	3.34	3.21	0.427
6	2.29	4.23	0.399	0.418	3.50	3.39	0.400
local attachment networks							
3	1.58	2.02	0.686	0.488	2.38	3.05	0.725
4	1.86	2.74	0.607	0.459	2.58	3.18	0.633
5	2.08	3.51	0.535	0.425	2.78	3.35	0.562
6	2.29	4.28	0.478	0.400	3.04	3.50	0.490

The probability $D(t')$ that a node i is in the network at time t falls exponentially with its age t' . In general, the degree distribution is given by

$$p(k) = \frac{1}{N} D(t'_k) \left| \frac{\partial \langle k(t') \rangle}{\partial t'} \right|_{t'=t'_k}^{-1}, \quad (23)$$

where t'_k is the solution of the equation $\langle k(t'_k) \rangle = k$ and $|\partial \langle k(t') \rangle / \partial t'|_{t'=t'_k}^{-1}$ is approximately the number of nodes with degree k (neglecting node deletion). With (10) we get

$$\frac{\partial \langle k(t') \rangle}{\partial t'} = \frac{m-s}{s} \frac{\langle k(t') \rangle}{N} \quad \text{and} \quad \frac{m-s}{s} \frac{t'}{N} = \ln \frac{k-c_1}{c_2}. \quad (24)$$

Then (assuming $k > c_1$) the degree distribution is given by

$$p(k) = \frac{s}{m-s} c_2^{\frac{s}{m-s}} (k-c_1)^{-\frac{m}{m-s}} \quad (25)$$

and the degree distribution $p(k)$ obeys a power-law tail with the exponent

$$\gamma = \frac{m}{m-s}. \quad (26)$$

If the parameters s and γ are determined independently from the degree distribution $p(k)$ by numerical experiments for preferential attachment networks with a nonlinear attractiveness (9), we find a reasonable agreement between γ and the predicted exponent (26), see Table I.

3.3.2. Local attachment

Preferential attachment with nonlinear attractiveness was introduced as an approximative description for our original local attachment model. It takes into account correlations on the

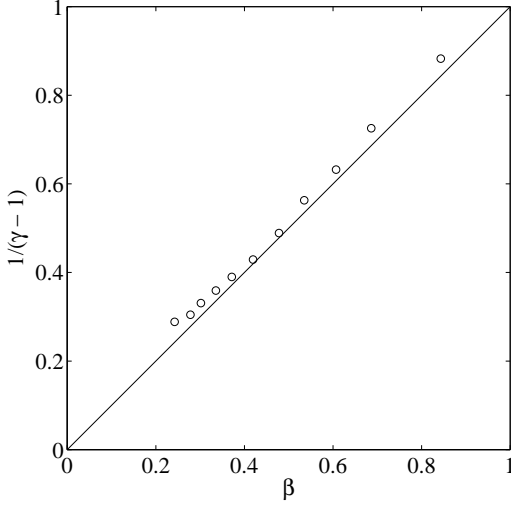


Figure 8: Scaling relation between exponents β and γ of local attachment networks under churn. Simulations have been carried out with networks of $N = 100\,000$ nodes with mean degree $2 \leq m \leq 12$.

level of the nonlinear mean attractiveness (6) that is observed in local attachment networks under churn.

In this way our simplified model is able to cover some important qualitative features of local attachment networks under churn. It accounts for the nonlinear mean attractiveness, for the exponential grow of $\langle k(i, t) \rangle$ (see Figure 7), and for the power-law degree distribution $p(k)$. Moreover, the combination of (21) and (26) gives the scaling relation

$$\beta = \frac{1}{\gamma - 1} \quad (27)$$

for preferential attachment networks under churn with nonlinear attractiveness. In fact, we find numerically that the universal scaling relation (27) is approximately fulfilled for nonlinear preferential attachment networks under churn as well as for local attachment networks under churn, see Figure 8 and Table I.

On the other hand, local attachment networks have higher order correlations that are not taken into account by the simplified model of nonlinear preferential attachment. As a consequence, not all results of section 3.3.1 hold also for our original model of local attachment networks under churn. For example numerical results show that the equations (21) and (26) that relate the mean attractiveness s and the mean degree m to the exponents β and γ are not fulfilled in the case of our original local attachment networks, see Table I. Local attachment networks and nonlinear preferential attachment networks with the same mean degree and the same mean attractiveness have different exponents β and γ .

4. Growing local attachment networks

In this section we consider a network growth process without churn that utilizes local attachment as a network-joining

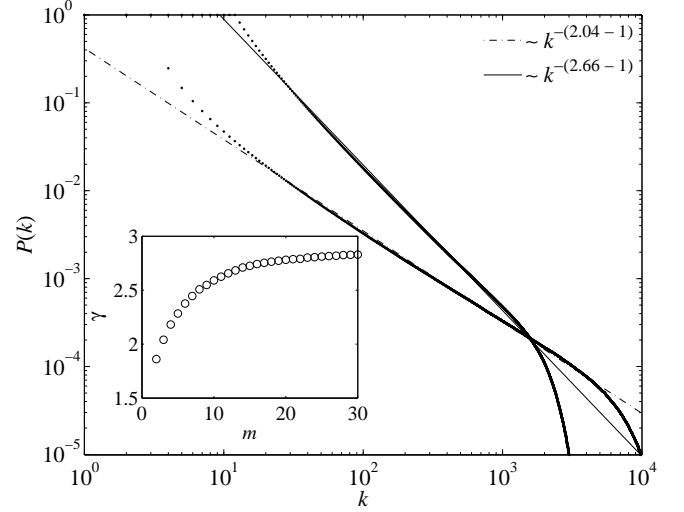


Figure 9: Cumulative degree distribution $P(k) = \sum_{i \geq k} p(i)$ of a growing local attachment network and a fit of this distribution to a power-law $P(k) \sim k^{-(\gamma-1)}$, which corresponds to $p(k) \sim k^{-\gamma}$. In the inset: Exponent γ of the power-law degree distribution as a function of the number m of edges that connect a new node to the network. Results are obtained for growing networks generated by the local attachment rule with $t = N = 100\,000$ nodes and $m = 3$ (dash-dot line) and $m = 12$ (dashed line).

protocol as introduced in section 2. The growth process starts from some initially-given network without isolated nodes, e. g. a network of $m + 1$ fully connected nodes or some other small connected network, and constructs a network that has in the limit of infinite large networks a mean degree $2m$ independently of the initial configuration.

4.1. Degree distribution

Numerically we find that our local attachment growth model results in networks with degree distributions that exhibit a power-law over a certain range of degrees, see Figure 9. The exponential cutoff of the degree distribution is a consequence of the finite size of the network. The exponent γ grows monotonically with parameter m , see the inset of Figure 9.

As in the case of the fixed-size network model in section 2 the origin of the power-law distribution of the local attachment growth model can be understood by considering the mean attractiveness. We find numerically that a node with degree $k \geq m$ attains under the dynamics of local attachment an average attractiveness that is given by the nonlinear function

$$\langle A(k) \rangle = k - k^*. \quad (28)$$

It can be shown analytically that network growth by preferential attachment with an attractiveness $A(k) = k - k^*$ gives networks with a power-law degree distribution with exponent $(3 - k^*/m)$ [22]. For our local attachment model the offset k^* can be measured numerically. The value $(3 - k^*/m)$ and the empirical exponent γ that is found by fitting the degree distribution to a power-law agree very well, see Figure 10.

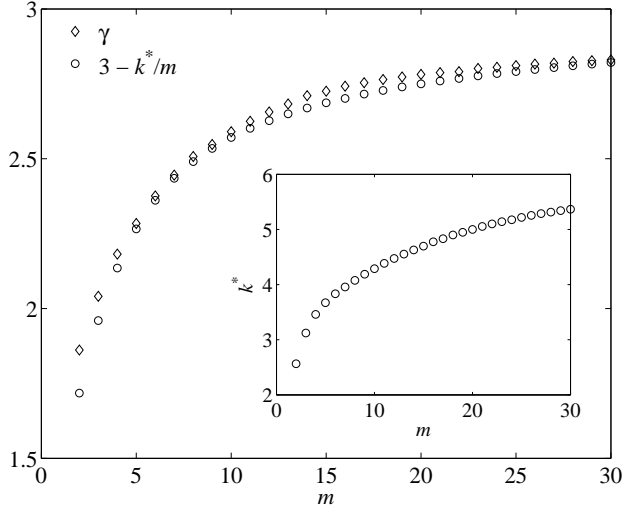


Figure 10: Local attachment procedure assigns effectively to each node an attractiveness proportional to $k_i - k^*$. The inset shows the offset k^* as a function of m . The diagram in the main figure shows the exponent γ of the degree distribution and $(3 - k^*/m)$.

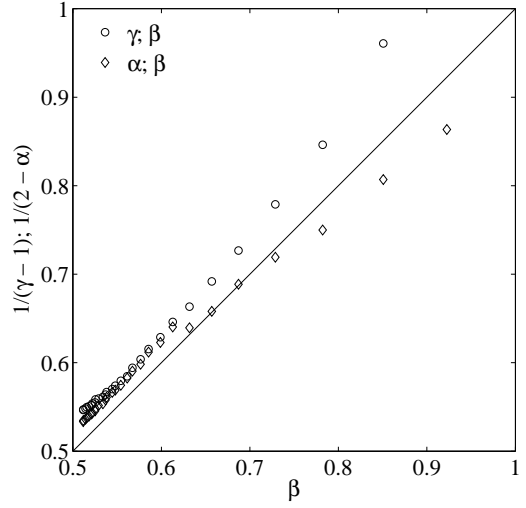


Figure 12: The scaling exponents α and β , and γ and β of growing local attachment networks satisfy approximately the relations $\beta = 1/(\gamma - 1)$, and $\beta = 1/(2 - \alpha)$ respectively. Each exponent has been calculated by averaging over more than 100 networks with $N = 100000$ nodes.

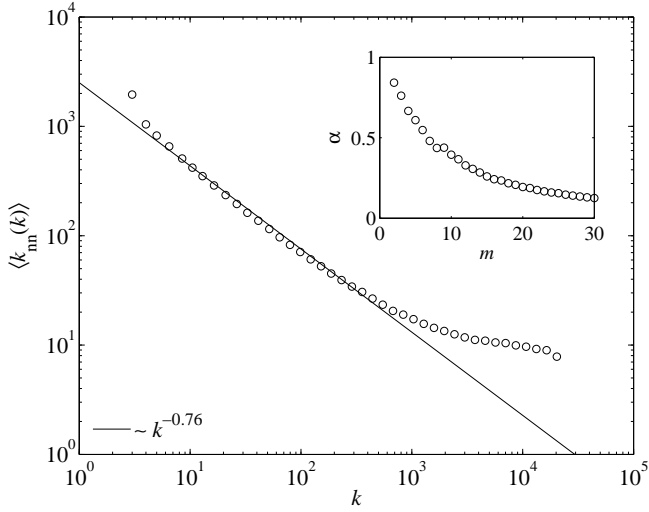


Figure 11: Distribution $\langle k_{nn}(k) \rangle$ of the mean degree of the neighboring nodes of a node with degree k for a growing local attachment network with $N = 100000$ nodes and $m = 3$. Inset: Exponent α of the power-law distribution $\langle k_{nn}(k) \rangle \sim k^{-\alpha}$ of the mean degree of the neighbors of a node with degree k as a function of the number m of edges that connect a new node to the network.

For local attachment networks without node removal the exponent γ tends to be smaller than for local attachment networks under churn. Node removal makes it harder to gain a large number of neighbors and therefore the degree distribution lacks nodes with very high degree in local attachment networks under churn.

4.2. Scaling relations

Local attachment growth networks can be characterized by further quantities that follow a power-law distribution. The mean degree $\langle k_{nn}(k) \rangle$ of the neighboring nodes of a node with degree k follows a power-law as well, see Figure 11. The exponent of this power-law $\langle k_{nn}(k) \rangle \sim k^{-\alpha}$ is negative and as a consequence nodes with large degree tend to be connected to nodes having a small degree and vice versa. This can be interpreted as a strong sign for disassortative correlations in the probability distribution $p(k, k')$ that a randomly chosen edge connects two nodes of degree k and k' . Furthermore, α decreases with the parameter m , that means the strength of the correlations depends on the density of the network, see the inset of Figure 11.

The mean degree $\langle k(i, t) \rangle$ of a node that has entered the network at time $t = i$ grows for $t \geq i$ following a power-law $\langle k(i, t) \rangle \sim (t/i)^\beta$. Numerically we find that the exponents γ and β are approximately connected via the relation

$$\beta = \frac{1}{\gamma - 1}, \quad (29)$$

and the exponents α and β follow a scaling relation as well, namely

$$\beta = \frac{1}{2 - \alpha}, \quad (30)$$

see Figure 12. In [1] it is argued that the scaling relations (29) and (30) should be fulfilled by any growing network where nodes of degree k have an attractiveness $A(k) = k - k^*$.

Note that degree distributions $p(k)$ and $\langle k(i, t) \rangle$ are related by the same scaling relations (27) and (29) in both variants of local attachment (nongrowing and growing). But the character of $\langle k(i, t) \rangle$ is quite different in the two cases, because for

growing networks β is the exponent of a power-law, whereas for constant size networks under churn β characterizes an exponential law.

5. Conclusions

We have introduced local attachment as an universal node-joining protocol that does not require global information. Nodes joining a network by local attachment incorporate only *local* information about the network topology to produce stable *global* properties. We investigated these global properties of local attachment networks under churn as well as of growing local attachment networks.

In both types of networks the degree distribution and other quantities follow a power-law. The exponents of these power-laws fulfill simple scaling relations.

The local attachment procedure induces disassortative cor-

relations between the degrees of neighboring nodes and the emergence of the power-law degree distribution for local attachment networks under churn is driven by these correlations and by statistical fluctuations.

Furthermore, correlations generate a nonlinear node attractiveness profile. The qualitative features of local attachment networks can be described by an effective model of preferential attachment that takes into account the nonlinear node attractiveness profile.

Acknowledgments

This work has been sponsored by the European Community's FP6 Information Society Technologies programme under contract IST-001935, EVERGROW. We thank the Institute for Theoretical Physics at Magdeburg University for providing computing time.

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